

**Arbitrary-Order Hermite Generating Functions
for Coherent and Squeezed States**Michael Martin Nieto¹*Theoretical Division, Los Alamos National Laboratory
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ABSTRACT

For use in calculating higher-order coherent- and squeezed- state quantities, we derive generalized generating functions for the Hermite polynomials. They are given by $\sum_{n=0}^{\infty} z^{jn+k} H_{jn+k}(x)/(jn+k)!$, for arbitrary integers $j \geq 1$ and $k \geq 0$. Along the way, the sums with the Hermite polynomials replaced by unity are also obtained. We also evaluate the action of the operators $\exp[a^j(d/dx)^j]$ on well-behaved functions and apply them to obtain other sums.

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1 Introduction

In calculations involving coherent and squeezed states, infinite sums involving Hermite polynomials are extremely useful. Even so, knowledge of such sums is limited. For example, if we define the generalized generating functions for the Hermite polynomials as

$$G(j, k) = \sum_{n=0}^{\infty} \frac{z^{jn+k} H_{jn+k}(x)}{(jn+k)!} , \quad (1)$$

for arbitrary integers $j \geq 1$ and $k \geq 0$, it is only the specific generating functions

$$G(1, 0) = \exp[-z^2 + 2xz] , \quad (2)$$

$$G(2, 0) = \exp[-z^2] \cosh(2xz) , \quad (3)$$

$$G(2, 1) = \exp[-z^2] \sinh(2xz) , \quad (4)$$

that are well-known [1]. Physically, Eqs. (2)-(4) are very important as they describe the decomposition of ordinary, even, and odd coherent states, respectively, into the harmonic-oscillator number states [2].

Motivated by this situation, in this paper we will show that the general generating functions [3] $G(j, k)$ are given by

$$G(j, k) = \frac{1}{j} \sum_{l=1}^j \frac{\exp[-z^2 e^{i4\pi l/j}] \exp[2xz e^{i2\pi l/j}]}{[e^{i2\pi l/j}]^k} . \quad (5)$$

Along the way we will also obtain the sums

$$S(j, k) = \sum_{n=0}^{\infty} \frac{z^{jn+k}}{(jn+k)!} = \frac{1}{j} \sum_{l=1}^j \frac{\exp[ze^{i2\pi l/j}]}{[e^{i2\pi l/j}]^k} . \quad (6)$$

We will use the same techniques to evaluate

$$I_j = \exp \left[\left(a \frac{d}{dx} \right)^j \right] g(x) , \quad (7)$$

and apply it to obtain further sums involving Hermite polynomials.

2 The Sums $S(j, k)$

First note the special cases

$$S(1, 0) = e^z, \quad (8)$$

$$S(2, 0) = \cosh z = \frac{1}{2} [e^z + e^{-z}]. \quad (9)$$

A few other sums are known. In particular, $S(3, 0)$ can be obtained from the known formula for $S(3, 1)$: [4]

$$S(3, 0) = \left(\frac{d}{dz} \right) S(3, 1) = \left(\frac{d}{dz} \right) \left[\frac{1}{3} e^z - \frac{2}{3} e^{-z/2} \cos \left(3^{1/2} \frac{z}{2} + \frac{\pi}{3} \right) \right]. \quad (10)$$

With a little algebra, the result can be written in the form

$$S(3, 0) = \frac{1}{3} \left(\exp[ze^{i2\pi/3}] + \exp[ze^{i4\pi/3}] + \exp[z] \right). \quad (11)$$

But from Eqs. (8), (9), and 11) we can now guess what the answer for $S(j, 0)$ is. It is

$$S(j, 0) = \frac{1}{j} \sum_{l=1}^j \exp[ze^{i2\pi l/j}]. \quad (12)$$

The phase factors in the exponentials are the j th roots of unity. Therefore, in the sum of the Taylor series expansions of the exponentials, after the z^0 term, the terms proportional to z, z^2, \dots, z^{j-1} cancel to zero. The next non-zero term is the z^j term. Similarly for the higher-order terms.

This argument can be made precise by using the geometric series

$$\sum_{l=1}^j r^l = \frac{r(r^j - 1)}{r - 1}. \quad (13)$$

Taking $r = \exp[i2\pi/j], r = \exp[i4\pi/j], r = \exp[i6\pi/j], \dots$, gives the result.

Then, $S(j, k)$ of Eq. (6) is obtained by observing that

$$S(j, k) = \left(\int dz \right)^k S(j, 0). \quad (14)$$

3 The Generating Functions $G(j, k)$

Now we are ready to calculate the generating functions $G(j, k)$. In Eq. (1) for $G(j, k)$, substitute the definition [1]

$$H_n(x) = (-1)^n \exp[x^2] \frac{d^n}{dx^n} \exp[-x^2] . \quad (15)$$

One then has

$$G(j, k) = \exp[x^2] \left[\sum_{n=0}^{\infty} \frac{(-z \frac{d}{dx})^{j n + k}}{(j n + k)!} \right] \exp[-x^2] , \quad (16)$$

which from Eq. (6) is equal to

$$G(j, k) = \exp[x^2] \left[\frac{1}{j} \sum_{l=1}^j \frac{\exp[-z e^{i 2 \pi l / j} \frac{d}{dx}]}{[e^{i 2 \pi l / j}]^k} \right] \exp[-x^2] . \quad (17)$$

But now using the relation (see below)

$$\exp \left[a \frac{d}{dx} \right] g(x) = g(x + a) , \quad (18)$$

one obtains the final result for $G(j, k)$ given in Eq. (5).

4 The Operators $\exp[a^j (d/dx)^j]$

The techniques we have used above can be applied to other situations. For example, we may ask if the result of Eq. (18),

$$I_1 = \exp \left[a \frac{d}{dx} \right] g(x) = \sum_{n=0}^{\infty} \frac{a^n g^{[n]}(x)}{n!} = g(x + a) , \quad (19)$$

can be generalized to $j > 1$:

$$I_j = \exp \left[\left(a \frac{d}{dx} \right)^j \right] g(x) = \sum_{n=0}^{\infty} \frac{a^{j n} g^{[j n]}(x)}{n!} . \quad (20)$$

First consider the special case $j = 2$:

$$I_2 = \sum_{n=0}^{\infty} \frac{a^{2 n} g^{[2 n]}(x)}{n!} . \quad (21)$$

Using the doubling formula

$$\Gamma(n + 1/2) = \pi^{1/2} \frac{(2n)!}{2^{2n} n!} , \quad (22)$$

one obtains

$$\begin{aligned} I_2 &= \frac{1}{\pi^{1/2}} \sum_{n=0}^{\infty} \frac{(2a)^{2n}}{(2n)!} \Gamma(n + 1/2) g^{[2n]}(x) \\ &= \frac{1}{2\pi^{1/2}} \sum_{n=0}^{\infty} \frac{(2a)^n}{n!} \Gamma\left(\frac{n+1}{2}\right) [g^{[n]}(x) + (-1)^n g^{[n]}(x)] \\ &= \frac{1}{2\pi^{1/2}} \int_0^{\infty} \frac{ds}{s^{1/2}} e^{-s} \sum_{n=0}^{\infty} \frac{(2as^{1/2})^n}{n!} [g^{[n]}(x) + (-1)^n g^{[n]}(x)] \\ &= \frac{1}{2\pi^{1/2}} \int_0^{\infty} \frac{ds}{s^{1/2}} e^{-s} [g(x + 2as^{1/2}) + g(x - 2as^{1/2})] . \end{aligned} \quad (23)$$

In the above, the use of $(-1)^n$ allowed us to cancel out the odd derivatives as we changed the sum from second-order terms to first-order terms. The use of the integral representation of the Γ function then led to the final result.

Note that by the change of variables

$$s = (y - x)^2 / 4a^2 , \quad (24)$$

our result can be put in the form of the known result [5]

$$I_2 = \frac{1}{(4a^2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{(y-x)^2}{4a^2}\right] g(y) dy . \quad (25)$$

(This formula will be useful in obtaining further sums in the next section.)

The way to generalize to arbitrary $j \geq 2$ is now, *in principle*, clear. First one uses the multiplication formula (which is the generalization of the doubling formula) in the form

$$\frac{1}{n!} = \frac{j^{jn+1/2}}{(2\pi)^{(j-1)/2}} \frac{\prod_{k=1}^{j-1} \Gamma(n + k/j)}{(jn)!} . \quad (26)$$

Then, just as before, the use of $\exp[i2\pi l/j]$ will allow the cancelation of undesired terms in the complete sum over n . Here those terms are the $g^{[p \neq nj]}$. When all is done,

the end result is

$$I_j = \frac{1}{(2\pi)^{(j-1)/2} j^{1/2}} \int_0^\infty \frac{dx_1 e^{-x_1}}{x_1^{1/j}} \int_0^\infty \frac{dx_2 e^{-x_2}}{x_2^{2/j}} \cdots \int_0^\infty \frac{dx_{j-1} e^{-x_{j-1}}}{x_{j-1}^{(j-1)/j}} \sum_{l=1}^j g\left(x + ja(x_1 x_2 \dots x_{j-1})^{1/j} e^{i2\pi l/j}\right). \quad (27)$$

This result, although in closed form, is usually too complicated to yield results in terms of elementary functions. For instance, even for $j = 3$, changing variables to $x_i = y_i^3$ yields

$$I_3 = \frac{9}{(2\pi)3^{1/2}} \int_0^\infty dy_1 y_1 e^{-y_1^3} \int_0^\infty dy_2 e^{-y_2^3} \sum_{l=1}^3 g\left(x + 3ay_1 y_2 e^{i2\pi l/3}\right). \quad (28)$$

But there is also another point that must be mentioned. In Ref. [6], it was observed that naive higher-order squeeze operators of the form $\exp[z_j a^{\dagger j} - z_j^* a^j]$, $j > 2$, are not Gaussian-integrable operators. Therefore, the convergence of the I_j , $j > 2$, depends on the form of $g(x)$.

Finally, for $j > 3$, Baker [7] has observed that using the Hankel contour integral for $1/\Gamma$ will lead to a formally simpler answer. Starting from Eq. (20), one has

$$I_j = \sum_{n=0}^{\infty} \sum_{l=1}^j \frac{a^n e^{i2\pi l n/j} g^{[n]}(x)}{j [(n/j)]!}, \quad (29)$$

where $[(n/j)]!$ is not clearly defined for n not a multiple of j . However, because of the sum over l , only the n -terms which are multiples of j will sum to non-zero. Therefore, $[(n/j)]!$ can be replaced by $1/\Gamma(1+n/j)$, which in turn can be replaced by the Hankel contour integral. Unity in the form of $\Gamma(n+1)/n!$ can be inserted, and $\Gamma(n+1)$ replaced by its integral representation. Then, a little algebra yields

$$I_j = \frac{1}{j2\pi i} \int_C \frac{dte^{-t}}{t} \int_0^\infty ds s e^{-s} \sum_{l=1}^j g\left(x + ast^{-1/j} e^{i\pi(2l-1)/j}\right). \quad (30)$$

C is the contour starting just above the x -axis at $(\infty, i\epsilon)$, coming in and encircling the origin counterclockwise, going out to $(\infty, -i\epsilon)$, and closing. It is straightforward to demonstrate that I_1 and I_2 of Eqs. (19) and (23) are special cases.

5 Discussion

With our methods, it is possible to obtain further quantities. Sums of the type

$$K(j, k, p, q) = \sum_{n=0}^{\infty} \frac{z^{jn+k} H_{jn+k}(x)}{(pn+q)!} \quad (31)$$

become amenable to analysis, as well as even more complicated sums.

For example, one can immediately write

$$\begin{aligned} K(2, 0, 1, 0) &= \exp[x^2] \exp \left[\left(-z \frac{d}{dx} \right)^2 \right] \exp[-x^2] \\ &= \frac{\exp[x^2]}{(4z^2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp \left[-\frac{(y-x)^2}{4z^2} \right] \exp[-y^2] dy \\ &= \frac{1}{[1+4z^2]^{1/2}} \exp \left[\frac{4z^2 x^2}{1+4z^2} \right]. \end{aligned} \quad (32)$$

This describes the squeezed ground state. Also, one has

$$\begin{aligned} K(2, 1, 1, 0) &= \exp[x^2] \left(-z \frac{d}{dx} \right) [K(2, 0, 1, 0) \exp[-x^2]] \\ &= \frac{2zx}{[1+4z^2]^{3/2}} \exp \left[\frac{4z^2 x^2}{1+4z^2} \right]. \end{aligned} \quad (33)$$

Adding these two yields the known sum [8]

$$\sum_{n=0}^{\infty} \frac{z^n H_n(x)}{[[n/2]]!} = \frac{1+2zx+4z^2}{[1+4z^2]^{3/2}} \exp \left[\frac{4z^2 x^2}{1+4z^2} \right], \quad (34)$$

where in the above $[[\cdot]]$ denotes the greatest integer function.

Acknowledgements

We wish to thank George Baker, Jr. and Jim Louck for their very helpful observations and comments. The work of MMN and DRT was supported by the US Department of Energy and the Natural Sciences and Engineering Research Council of Canada, respectively

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